Final Exam : Waves & Optics

29 January 2018, 9:00-12:00

- Put your name and student number on each answer sheet.
- Answer all questions short and to the point, but complete; write legible.
- Final point grade = 9(total number of points/30) + 1

1. **Muzo (9 points)**
   Consider the paper *Muzo - State of the Art Vibration Monitoring System*
   The proposed device described in the article is claimed to be able to create a noise-free environment by using anti-sound generated by the Muzo device attached to a window or other hard surface. The device will record the incoming sound through a microphone, and will generate anti-sound by producing vibrations in the surface it is attached to, which will in turn generate sound in the surrounding air. Here, you may assume that Muzo acts as a point source.

   a) Explain the principle behind the functioning of Muzo assuming a single point-source of harmonic noise at position $\vec{R}_N$ with frequency ω, amplitude A_N, and initial phase ϕ_0, and with a Muzo device placed at position $\vec{R}_M$. Do this for an observer at a fixed position $\vec{R}_O$. Include a full and complete description of the various waves in your answer. **(3 points)**
   The functioning is based on the concept of addition of waves, in particular destructive interference. A point source at location R_N will create a spherical wave that produces an amplitude at location R given by

   $$A(R) = \frac{A_N}{|R - R_N|} e^{i[\omega t - k|R-R_N| + \phi_0]}.$$

   Similarly for the wave emitted by MuZo,

   $$A'(R) = \frac{A_M}{|R - R_M|} e^{i[\omega t - k|R-R_M| + \phi_M]}.$$

   Cancellation of noise at location R_O is obtained when $A(R_O) + A'(R_O) = 0$. The wave emitted by MuZo is based on the incoming wave at the location of MuZo, so $A(R_M)$. In it's simplest approximation $A_M = -|A(R_M)|$.

   b) For optimal noice reduction it is necessary that both the amplification (ratio of measured and emitted amplitudes) and phase-shift (difference between measured and emitted phase) of Muzo are adjustable by the user. Explain why. **(3 points)**
   Since the observer, MuZo and the noise source are not necessarily in a line, the distance between the noise source and the observer is not simply given by the sum of the distances from the noise source to MuZo and from MuZo to the observer. Hence the resulting amplitude attenuation and phase differences need to be taken into account.

   c) The following situation may be analyzed as a 1D problem. Assume that Muzo is attached to a window on one side of a long room of 10 m length and that the other side of the room consists of a concrete wall. Both the wall and the window almost perfectly reflect incoming sound waves. Explain that there is a risk that Muzo will start to “ring”, which is the situation where it will try to generate certain fixed frequencies with a huge amplitude.

Given that the speed of sound is 340 m/s, which is the lowest frequency at which this could occur? Recall that the microphone of Muzo is positioned a little away from the surface of the window. **(3 points)**
The long room, with its near perfect reflection, starts to act as a Fabry-Perot cavity, with standing waves inside. All wavelengths for which $n\lambda \simeq 2L$ can be maintained. How closely this relation needs to be met depends on the reflectivity, and thus the (coefficient of) Finesse. If the microphone of MuZo is positioned a little away from the surface, it may "hear" the standing wave. It will try to cancel this wave by emitting anti-sound at the window, which is precisely a node of the wave and hence has no effect for any amplitude. The lowest frequency for which this is possible corresponds to the longest possible wavelength, which is $\lambda = 2L = 20\,\mathrm{m}$. So $\nu = \frac{v}{\lambda} = \frac{340\,\mathrm{m/s}}{20\,\mathrm{m}} = 17\,\mathrm{Hz}$ (or $\omega = 107\,\mathrm{Hz}$).

2. **Bunches (6 points)**
consider the paper *Bunch length measurements using a Martin-Puplett interferometer at the Tesla test Facility LINAC.*

   a) The transmission of a two-arm interferometer, such as the Martin-Puplett and Michelson-Morley ones, varies when varying the length of one of the arms. Explain why it is not possible to find the point where both arms have equal length using a monochromatic light-source. **(3 points)**
   For a mono-chromatic light source the light has only a single frequency, and therefore the interference pattern is harmonic, *i.e.* repetitive. Any pathlength difference equal to an integer multiple of the wavelength will result in an interference maximum. For poly-chromatic light, which has multiple wavelengths, these maxima will occur at different pathlength differences. Only for a pathlength difference of zero these maxima coincide for all wavelengths simultaneously.

   b) The purpose of the Tesla facility is to measure the (longitudinal) shape of short electron beam bunches that repeatedly pass the radiator. When the electrons pass the radiatior they generate an electric field proporional to the instantaneous beam intensity, $E(t) \propto I(t)$. Show how the corresponding (discrete) frequency spectrum of the electric field can be calculated. How is the frequency spacing related to the repetition period of the electron beam? How does the shape of the bunches show up? **(3 points)** .
   These pulses are a case of anharmonic repetitive waves for which we can use Fourier-analysis. The frequency spectrum is calculated (in general) from $\tilde{E}(\omega) = \int E(t)e^{-i\omega t}dt$. In our case the electric field is repetitive, $E(t + nT) = E(t)$ with T the repetition time. This results in a discrete frequency spectrum, with frequency spacing $\Delta\omega = 2\pi/T$. Hence, $\tilde{E}(\omega) = A(\omega) \cdot \delta(\omega - n\frac{2\pi}{T})$. The envelope $A(\omega)$, which gives the amplitude of the succesive peaks in the frequency spectrum, is due to the shape of a single pulse, $A(\omega) = \int E(t)e^{-i\omega t}dt$ for $T \to \infty$.

3. **Interferometer (12 points)**
consider the paper *The first direct observation of gravitational waves.*
You are constructing your own Michelson-Morley interferometer setup. Instead of the normal setup which uses a partially reflective mirror, you use a polarizing beamsplitter (which transmits the vertical component of the electric field, and reflects the horizontal component by 90°). All other components are the same as in a standard Michelson-Morley setup.

   a) You find that no light is transmitted to the detector at all, even if you vary the arm lengths. Explain why this is. (If you do so via a sketch, make sure to label all essentials!)

(3 points)
All light that is transmitted by the beam splitter will also be transmitted on the way back from the mirror. All the light that is reflected by the beam splitted is also reflected on the way back. Hence non of the light will be directed towards the detector.

b) By placing a bi-refringent optical element in each of the arms you manage to transmit light to the detector. Explain which properties this element must have. **(3 points)** .
These elements must rotate the linear polarization by 90°. The light passes the element twice (once going forward, once going back). A $\lambda/4$ plate, when traversed twice, will lead to a phase shift of 180 degrees between the two orthogonal components of an EM wave. When placed under an angle of 45 degrees w.r.t. the plane of linear polarization this will cause a rotation of that plance by 90 degrees. In that case the originally vertically polarized beam will become horizontally polarized and be transmitted by the cube. Similarly for the horizontally polarized beam, which will now be reflected. Both beams will hit the detector, and not be reflected back to the light source.

c) When you vary the length of one of the arms, you still do not observe interference, even when you use a laser. Why is that? **(3 points)**
The light following the two paths have different polarizations. Hence no interference can occur.

d) In the LIGO article it is stated that "To improve sensitivity, each arm of the basic Michelson interferometer contains a Fabry-Perot cavity that stores the photons while the light bounces back and forth between the test mass mirrors, thus increasing the duration of the gravitational wave's interaction with the light. The effect on the light phase is a factor 300, an effective increase of the arm length to over 1000 km." Show how you can use this effective increase to make an estimate of the reflectance of the mirrors that form the cavity. **(3 points)**
Several approaches are acceptable
(1) Via Finesse: the sensitivity to variations in the reflectance (beam has to go back in the direction of the source/beam splitter) relative to the incident irradiance are proportional to the (coefficient of) Finesse, which in turn depends on the reflectance of the mirrors. As an irradiance variation is the signal of LIGO, its sensitivity is thus proportional to F. Irradiance variation due to a length variation of the arm is proportional to the arm's length. Hence F and relative length are equivalent (except at precisely the minimum and maximum). See also lect. notes pg. 61.
(2) Probabilistic: reflectivity r is the chance for a photon to remain in the cavity when it hits the mirror. The survival probability for a photon inside the cavity after many hits on the mirror $R(N) = r^N$ is never zero, just like the intensity inside the cavity is never zero (only in the conintuum limit). From that we can calculate the average number of bounces as $\hat{N} = \sum Nr^N / \sum r^N = r/(1-r)$. For large r this is approximately $\hat{N} = 1/r$. Since $N = 300$ is gives $r \simeq 1 - 1/300$.
(3) Every time that the lightbeam that bounces back and forth between the mirrors of the cavity a small fraction of the beam leaves the cavity. The amplitude of each succesive emission contains a common factor, namely the transmission t of the exit mirror (the light should exit the cavity), the reflectance r_{back} of the mirror on the other side (the light needs to be reflected at least once), and a factor ρ that is determined by the number of times the beam has bounced back and forth:

$$\rho(N) = (r_{front} r_{back})^N = rr^N$$

Here N is the number of times the beam bounces *back and forth* in the cavity. The amplitude of the emitted light after N bounces is $E(N) = tr_{back}E_0\rho(N)$. So the amplitude of the emitted light after N bounces is $E(N) = E_0\rho(N)$. In addition to modification of the amplitude also, more importantly, the phase is affected. If the length of the cavity is D, each back and forth bounce results in an additional phase shift of $\delta\phi = 2kD$, so after N bounces $\Delta\phi = N\delta\phi = 2NkD$. In complex notation, each additional wave escaping from the cavity is modified by a complex factor

$$f(N) = \rho(N)e^{i\Delta\phi} = rr^N\, e^{i2NkD} = \left(rr\, e^{i2kD}\right)^N.$$

The total complex amplitude is then given by

$$E_{tot} \propto \sum_{N=0}^{\infty} f(N) = \frac{1}{1 - rr\, e^{i2kD}}.$$

This has a norm and a phase.

4. **Micro-fossils (3 points)**
A team of paleo-archeologists wants to scan a transparent piece of amber to search for embedded micro-fossils. They propose to shine a laser-beam through the amber and measure the transmission while moving the amber. A fossil would block the light and thus cause a shadow. For better resolution it is proposed to make the diameter of the laser beam as small as possible by shining it through a narrow aperture. Explain why using an extremely small aperture would actually *worsen* the resolution instead of improving it. The amber is very far away from the aperture relative to its size. Use that the senstivity area of the light detector is a lot larger than the size of the fossil.
Ref. Lect. Notes Pg. 68. When shining light through an aperture diffraction occurs. In the Fraunhofer limit, where the light source and observer are far from the aperture (as we have here), this leads to an angular spread and thus an image of the aperture at the location of the observer which is larger than the aperture itself. The defining parameter is $\beta = k\,a\,\sin\theta$. For the same k (thus wavelength), a smaller aperture diameter a thus leads to a larger $\sin\theta$ to get the same intensity drop-off. Note that diffraction plays a prominent role only when a becomes relatively small compared to the wavelenth, so only for extremely small sizes.